

# Models for the $k$ -metric dimension

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## Abstract

For an undirected graph  $G = (V, E)$ , a vertex  $\tau \in V$  separates vertices  $u$  and  $v$  (where  $u, v \in V, u \neq v$ ) if their distances to  $\tau$  are not equal. Given an integer parameter  $k \geq 1$ , a set of vertices  $L \subseteq V$  is a feasible solution if for every pair of distinct vertices,  $u, v$ , there are at least  $k$  distinct vertices  $\tau_1, \tau_2, \dots, \tau_k \in L$  each separating  $u$  and  $v$ . Such a feasible solution is called a *landmark set*, and the  $k$ -metric dimension of a graph is the minimal cardinality of a landmark set for the parameter  $k$ . The case  $k = 1$  is a classic problem, where in its weighted version, each vertex  $v$  has a non-negative weight, and the goal is to find a landmark set with minimal total weight. We generalize the problem for  $k \geq 2$ , introducing two models, and we seek for solutions to both the weighted version and the unweighted version of this more general problem. In the model of all-pairs (AP),  $k$  separations are needed for every pair of distinct vertices of  $V$ , while in the non-landmarks model (NL), such separations are required only for pairs of distinct vertices in  $V \setminus L$ .

We study the weighted and unweighted versions for both models (AP and NL), for path graphs, complete graphs, complete bipartite graphs, and complete wheel graphs, for all values of  $k \geq 2$ . We present algorithms for these cases, thus demonstrating the difference between the two new models, and the differences between the cases  $k = 1$  and  $k \geq 2$ .

## 1 Introduction

The problem of finding a landmark set or a resolving set of a graph was studied in a number of papers [19, 11, 2, 8, 16, 15, 5, 18, 17, 13, 10, 12]. In this problem, one is interested in finding a subset of vertices  $L$  of an undirected graph  $G = (V, E)$ , such that the ordered list of distances of a vertex  $u$  to the vertices of  $L$  uniquely determine its identity. That is, the set  $L \subseteq V$  should be such that for any  $u, v \in V$  ( $u \neq v$ ), there exists  $\tau \in L$  such that  $d(u, \tau) \neq d(v, \tau)$  (letting  $d(x, y)$  denote the number of edges in a shortest path between  $x$  and  $y$ ). We say that such a vertex  $\tau$  separates  $u$  and  $v$ , and a set  $L$  satisfying the property that for every pair of distinct graph vertices it contains a vertex that separates them, is called a landmark set. Since for any  $\ell \in L$ ,  $d(\ell, \ell) = 0$  and  $d(\ell, x) > 0$  for any  $x \neq \ell$ , every landmark  $\ell$  separates itself from any vertex  $v \neq \ell$ , and the requirement that a separating vertex  $\tau$  will exist for any  $u, v \in V \setminus L$  ( $u \neq v$ ) is equivalent to the requirement that  $\tau$  will exist for any distinct pair of vertices of  $V$ . For a given graph, the algorithmic problem of finding a landmark set of minimum cardinality is called the metric dimension problem (and this minimum cardinality is the metric dimension). In the weighted case, a non-negative rational weight is given for each vertex by a function  $w : V \rightarrow \mathbb{Q}^+$ . For a set  $U \subseteq V$ ,  $w(U) = \sum_{v \in U} w(v)$  is the total weight of vertices in  $U$ , and the goal is to find a landmark set  $L$  with the minimum value  $w(L)$ , where its weight is called the weighted metric dimension.

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One limitation of this model in large networks is the frequent failures of vertices. When a failure occurs at a landmark vertex, this can harm the identification process for other vertices. We generalize the metric dimension problem and require that  $L$  will contain  $k$  or more separating vertices for every pair of distinct vertices for a given parameter  $k \geq 2$ . This problem is called the  $k$ -metric dimension problem, and its weighted variant is called the weighted  $k$ -metric dimension problem. There can be two types of requirements for landmark sets. The first one is to separate (by  $k$  or more separations) any pair of distinct vertices of  $V$ , and the second one considers separations for any pair of distinct vertices of  $V \setminus L$ . We consider both these models in this work. We refer to the first variant as the all-pairs model (AP), and the second variant is called the non-landmarks model (NL). We will demonstrate the differences between landmark sets for the case  $k = 1$  and for larger values of  $k$ , and the differences between solutions for the two models. This is done using several graph classes with simple structures. These graph classes are path graphs, complete graphs (cliques), complete bipartite graphs, and complete wheels<sup>1</sup>.

Given the input graph  $G = (V, E)$ , let  $n = |V|$  be the number of its vertices. The set of landmark sets of  $G$  for the parameter  $k$  ( $k \geq 2$ ) is denoted by  $LS_k^M(G)$ , where  $M = AP$  or  $M = NL$  (we use  $M$  when we discuss a specific model and not some general facts that are not model dependent). The minimum cardinality of any landmark set, which is the cost of an optimal solution for the unweighted problem, is denoted by  $md_k^M(G)$ . A set  $Q \in LS_k^M(G)$ , such that  $w(Q) \leq w(D)$  for any set  $D \in LS_k^M(G)$ , is a solution for the weighted problem, and  $wmd_k^M(G) = w(Q)$  denotes the weight of this solution. For a vertex  $v \in V$  we use the notation  $N_v(G) = \{u \in V : d(u, v) = 1\}$ , that is, all neighbors of  $v$ . In addition, for  $u, v \in V$  we use the notation  $N_{u,v}(G) = N_u(G) \cap N_v(G)$  for all the common neighbors of  $u$  and  $v$  in  $G$  and the notation  $Sp_{u,v}(L) = \{z \in L : d(u, z) \neq d(v, z)\}$  which denotes the set of all vertices in  $L \subseteq V$  that separate  $u$  and  $v$ . If there is no feasible solution, we will write  $md_k^M = \infty$  and  $wmd_k^M = \infty$ . This case is possible for AP, but for NL, as we will explain below, a feasible solution always exists.

By definition, for any connected undirected graph  $G$ , if  $L \in LS_k^{AP}(G)$ , then  $L \in LS_k^{NL}(G)$ . Thus, for any integer  $k \geq 2$ ,  $md_k^{NL}(G) \leq md_k^{AP}(G)$ , and  $wmd_k^{NL}(G) \leq wmd_k^{AP}(G)$  for any non-negative weight function  $w : V \rightarrow \mathbb{Q}^+$  on  $G$ 's vertices. Another obvious property is  $LS_{k+1}^M(G) \subseteq LS_k^M(G)$  for any  $k \geq 1$  and any model  $M$ . The next lemma will be used for analyzing landmark sets.

**Lemma 1** *Let  $L \subseteq V$ , and let  $u, v \in L$  be a distinct pair of vertices. Then  $u$  and  $v$  are separated by at least two vertices of  $L$ .*

**Proof.** Since  $u, v \in L$ ,  $d(u, u) = d(v, v) = 0$  and  $d(u, v) \neq 0$  since  $u \neq v$ , thus  $u$  and  $v$  separate each other. ■

In the body of the paper we assume  $n \geq 2$ . We now discuss the cases  $n = 1, 2$ , and several other simple cases where the solutions do not depend on the structure of the graph. The case  $n = 1$  is trivial, as in this case  $\emptyset$  is a valid landmark set for both models and any  $k \geq 1$ . For AP, if  $2 \leq n \leq k - 1$ , there are no feasible solutions, as a feasible solution must have at least  $k$  landmarks. In the case  $n = k = 2$ ,  $V$  is the unique landmark set as a landmark set for AP must have at least  $k$  vertices, and by Lemma 1. We will show in Section 2 that in the case  $n = k \geq 3$ , there is no feasible solution for AP. On the other hand, for NL, any subset of  $n - 1$  vertices is a feasible solution for any  $k$ , and thus a feasible solution always exists. We call such a solution *a trivial solution*. A

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<sup>1</sup>In an accompanying paper [1], we analyze additional graph classes with respect to the concepts that are introduced in this paper.

solution consisting of  $n$  vertices is also feasible, but we will never use such a solution for NL as a solution of  $n - 1$  vertices always has smaller cardinality and smaller weight. Any solution that is not trivial must consist of at least  $k$  vertices. Thus, for  $2 \leq n \leq k + 1$ , any optimal solution is a trivial solution. On the other hand, if  $n \geq k + 2$ , a solution consisting of at most  $k - 1$  vertices cannot be valid. Thus, in particular, if  $n = 2$ , then  $md_2^{AP} = 2$  and  $md_2^{NL} = 1$ , and for  $k \geq 3$  (and  $n = 2$ ),  $md_k^{AP} = \infty$  and  $md_k^{NL} = 1$ . For any graph  $G$ ,  $V \in LS_2^{AP}(G)$ , so a landmark set always exists in this case. However, there are graphs where there is no feasible solution even if  $n > k \geq 3$  (see Section 3.1). Given the cases that are completely resolved independently of the graph structure, in what follows, we will deal with  $n \geq 3$  and  $n \geq k + 1$  for AP (and we will prove that there is no solution for  $n = k \geq 3$ ), and with  $n \geq k + 2$  for NL.

As in [10], we are often interested in minimal landmark sets with respect to set inclusion, since any minimum weight landmark set must be such a set. Note that in many cases not every minimal landmark set (with respect to set inclusion) is a minimum cardinality landmark set. An algorithm for finding a minimum weight landmark set can test minimal landmark sets to find one of minimum weight (by enumerating them or via dynamic programming).

**Previous work.** As mentioned above, most previous studies dealt with the case  $k = 1$ . In [7], a variant where separation is defined using pairs of vertices was studied (see also [4]). That model is different from ours, and in particular, the condition for a pair of landmark vertices  $u, v$  to doubly separate a pair of graph vertices  $x, y$  is defined on all four vertices together, and it is required that  $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$ . Note that by definition,  $u, v$  cannot satisfy this if  $u = v$ , but  $u, v$  doubly separate  $u, v$ . A double landmark set  $S$  is a subset of  $V$  such that for any  $x, y \in V$ , there exist  $u, v \in S$  that doubly separate  $x$  and  $y$ . There is no particular relation between this condition and the conditions that we require for  $k \geq 2$  (neither for NL nor for AP), as can be seen in the following examples. Consider a path graph on  $n \geq 5$  vertices;  $v_1, v_2, \dots, v_n$  (in this order). In Section 2, it is shown that the subset  $X = \{v_2, v_3, \dots, v_{n-1}\}$  is a landmark set for  $k \leq n - 3$  (for both AP and NL). This is, however, not a double landmark set, as the  $d(u, v_1) - d(v, v_1) = d(u, v_2) - d(v, v_2)$  for any  $u, v \in X$  (see [7]). On the other hand, if the graph is a wheel with seven vertices  $c_1, \dots, c_7$  (where  $c_7$  is the central vertex, see Section 3), by the results of [7],  $c_1, c_3, c_5$  is a double landmark set, while it gives only one separation for  $c_7$  and  $c_2$ , and thus it is not a landmark set even for  $k = 2$  (for both AP and NL). This difference holds for larger wheels as well, as can be seen from the fact that the conditions that we present for landmark sets of wheels in Section 3 differ from the condition given in [7] for double landmark sets.

This metric dimension problem was introduced over forty years ago by Harary and Melter [11] and by Slater [19]. It was studied widely in the combinatorics literature, where exact values of the metric dimension or bounds on it for specific graph classes are obtained [2, 4, 5, 6]. The problem was also studied with respect to complexity in the sense that it was shown to be NP-hard in general [15], it was shown that it is hard to approximate [3, 13, 12], and it is even NP-hard for certain graph classes [9, 10]. On the other hand, it is polynomially solvable (often even for the weighted version) for many graph classes; in particular such classes include paths, cycles, trees, and wheels [19, 11, 15, 5, 18]. The main graph classes studied here are paths and wheels, which were studied in [15, 10] and [11, 18, 10], respectively. It was proved by Khuller, Raghavachari, and Rosenfeld [15] that the metric dimension (for  $k = 1$  and the unweighted case) of a path is 1, and was shown by Shanmukha, Sooryanarayana and Harinath [18] that the metric dimension is  $\lceil \frac{2n}{5} \rceil$  for a complete wheel with  $n \geq 8$  vertices (for  $n = 4, 5, 6$ , and 7 the metric dimension is equal to 3, 2, 2 and 3,

respectively). In the weighted case, it is shown in [10] that a minimum weight landmark set for a path consists of one or two vertices. For complete wheels, a condition on the positions of landmarks on the cycle of the wheel was stated [10] (see Section 3). Here, we define several conditions for the different cases (according to the different values of  $k$ , and the two models), where these conditions differ from that of the case  $k = 1$  in all cases. We also study complete graphs, for which any landmark set for  $k = 1$  consists of at least  $n - 1$  vertices [5], and complete bipartite graphs, where (if the graph is connected and  $n \geq 3$ ), any landmark set contains all vertices except for at most one vertex of each partition [5] (for  $k = 1$ ).

## 2 Paths

We start with path graphs. These graphs are typically easier to analyze than any other graph class. We will show that while in AP landmark sets for  $k \geq 2$  have similar structures to those of the case  $k = 1$ , in NL this is not the case.

### 2.1 All-pairs model (AP)

In this section, we let  $G = (V, E)$  be a path graph with  $n \geq 3$ . Let  $k \leq n - 1$ , and let  $V = \{v_1, \dots, v_n\}$ , where the vertices appear on the path in this order, i.e.,  $E$  consists of the edges  $\{v_i, v_{i+1}\}$  for  $1 \leq i \leq n - 1$ . The vertices  $v_1, v_n$  are called end vertices, while the other vertices are called internal vertices (and every path where  $n \geq 3$  has at least one such vertex). We will adapt the following property.

**Claim 2** *For  $k = 1$ ,  $md_1^{AP}(G) = 1$ , and a minimum cardinality landmark set consists of either  $v_1$  or  $v_n$  [15]. Any minimal (with respect to set inclusion) landmark set that is not  $\{v_1\}$  or  $\{v_n\}$  consists of exactly two internal vertices [10].*

**Proposition 3** *For  $n \geq 2$ ,  $md_2^{AP}(G) = 2$ , and  $md_k^{AP}(G) = k + 1$  for  $k \geq 3$ . Moreover, for  $k \geq 3$ , a set  $L \subseteq V$  is a minimal landmark set if and only if  $|L| = k + 1$ . For  $k = 2$ , a set  $L \subseteq V$  is a minimal landmark set (with respect to set inclusion) if and only if either  $L = \{v_1, v_n\}$  or  $|L| = 3$  and  $\{v_1, v_n\} \subsetneq L$ . A minimum cardinality or minimum weight landmark set can be found in time  $O(n)$ .*

**Proof.** We start with showing that any subset  $X$  of  $k + 1$  vertices is a landmark set. This holds as for any pair  $v_i, v_j \in V$  ( $i \neq j$ ), there is at most one vertex of equal distances to both of them. Specifically,  $v_{(i+j)/2}$  has equal distances to  $v_i$  and  $v_j$ , if  $i + j$  is even (and otherwise, if  $i + j$  is odd, then there is no such vertex). Thus, there are  $k$  or  $k + 1$  separation vertices in  $X$  for any pair  $v_i, v_j$ .

Recall that a landmark set for AP must contain at least  $k$  vertices. We show that a set  $X$  consisting of  $k$  vertices such that at least one of them is internal is not a landmark set. Assume that  $v_a \in X$ , where  $1 < a < n$ . The two vertices  $v_{a-1}$  and  $v_{a+1}$  have equal distances to  $v_a$ , and therefore they have at most  $k - 1$  separations. This shows that except for the case  $k = 2$ , any landmark set has exactly  $k + 1$  vertices. Since any subset of  $k + 1$  vertices is a landmark set for  $k \geq 2$ , we find that for  $k \geq 3$ , the class of minimal landmark sets is exactly the class of subsets of  $k + 1$  vertices. Finding a minimum weight such subset can be done in linear time by selecting  $k + 1$  vertices of minimum weights (breaking ties arbitrarily).

For  $k = 2$ ,  $\{v_1, v_n\}$  is in fact a landmark set, as the distances of graph vertices from  $v_1$  and from  $v_n$  are unique, and it is the only landmark set of minimum cardinality. Thus, any set of the form  $\{v_1, v_i, v_n\}$  for  $1 < i < n$  is not a minimal landmark set (with respect to set inclusion), while other sets consisting of three vertices are minimal landmark sets. Finding a minimum weight landmark set can be done as follows. Select the minimum weight solution out of the following three solutions: a minimum weight subset of three internal vertices, a minimum weight subset of two internal vertices plus a minimum weight vertex out of the end vertices, and finally, the third option is the solution  $\{v_1, v_n\}$ . These solutions can be computed in linear time as well. ■

We finish this section with showing that for any graph with  $n = k \geq 3$  there does not exist a landmark set.

**Claim 4** *For any graph with  $n = k \geq 3$ , there is no feasible solution for AP.*

**Proof.** Assume by contradiction that there is a landmark set  $L \subseteq V$  for a graph  $G$  with  $n = k \geq 3$ . Since  $n = k$ , in order to obtain  $k$  separations for any pair of distinct vertices, we find  $L = V$ , and every vertex of  $V$  must separate any pair of distinct vertices of  $V$ . As distances in the graph can take values in  $\{0, 1, \dots, n-1, \infty\}$ , the graph must be connected. This holds as in a disconnected graph, there exists a pair of vertices  $u, v \in V$  where  $d(u, v) = \infty$ , and in this case for any vertex  $x \in V$ ,  $x \neq u, v$  ( $x$  must exist as  $n \geq 3$ ), at least one of  $d(x, u) = \infty$  and  $d(x, v) = \infty$  must hold since otherwise  $d(u, v) < \infty$ , and either  $u$  does not separate  $x$  and  $v$  or  $v$  does not separate  $x$  and  $u$  (or both).

Since the graph is connected, given a vertex  $v \in V$ , there are  $n-1$  distinct values for the distances  $d(v, y)$  for  $y \in V$ , and there exists  $y'$  such that  $d(v, y') = n-1$ . The graph must be a path connecting  $v$  and  $y'$ . Let  $x' \in V \setminus \{v, y'\}$  (where  $x'$  must exist as  $n \geq 3$ ). Similar reasoning shows that there is a vertex  $y''$  ( $y'' \in V \setminus \{x'\}$ ) such that the graph is a path connecting  $x'$  and  $y''$ . As  $x \neq v, y'$ , we reach a contradiction. ■

## 2.2 Non-landmarks model (NL)

In this section we will show that  $md_k^{NL}(G) = k$  for any  $k \geq 2$  and  $n \geq k+2$ . Since any subset of  $k+1$  vertices is a landmark set for AP and therefore also for NL, we focus on subsets of  $k$  vertices. Any such landmark set is obviously minimal with respect to set inclusion. Given a set  $L \subseteq V$ , we say that a hole is a vertex of  $V \setminus L$ . A maximum length sequence of consecutive holes is called a gap (induced by  $L$ ). The length of a gap is defined to be the number of vertices in it. A gap can possibly contain an end vertex. As  $n > k$ , if  $|L| = k$ , there is at least one hole, thus at least one gap induced by  $L$ . Similarly, a vertex  $v_i \in L$  is called an anti-hole, and a sequence of consecutive vertices of  $L$  is called an anti-gap.

The following two lemmas deal with the case where  $L \subseteq V$  is a set of vertices that satisfies  $|L| = k$ .

**Lemma 5** *If  $L$  induces exactly one gap, then  $L$  is a landmark set. Thus,  $md_k^{NL} = k$ .*

**Proof.** Since there is just one gap,  $L = \{v_1, \dots, v_a, v_b, \dots, v_n\}$ , such that  $0 \leq a < b \leq n+1$  and  $a + (n - b + 1) = k$  (if  $a = 0$ , then  $v_1 \notin L$ , and if  $b = n+1$ , then  $v_n \notin L$ ). Consider the vertices of  $V \setminus L$ , that is,  $v_{a+1}, \dots, v_{b-1}$ . For any pair of vertices  $v_i, v_j$ , such that  $a < i < j < b$ , the distances to any vertex in  $L$  are distinct (as the path from  $v_j$  to any vertex  $v_{a'}$  with  $a' \leq a$  traverses  $v_i$ , and

the path from  $v_i$  to any vertex  $v_{b'}$  with  $b' \geq b$  traverses  $v_j$ ). As such a subset  $L$  must exist (for example,  $\{v_1, v_2, \dots, v_k\}$  is such a set), we have  $md_k^{NL} = k$ . ■

**Lemma 6** *If  $L$  induces at least two gaps, and at least one gap has at least two vertices, then  $L$  is not a landmark set.*

**Proof.** Assume that  $L$  induces two such gaps. Consider two consecutive gaps (where the set of vertices between them is an anti-gap) such that at least one of these gaps has two vertices. Let  $v_i$  be the last vertex of the first gap out of the two, and let  $v_j$  ( $j \geq i + 2$ ) be the first vertex of the second gap. Without loss of generality assume that  $v_{j+1}$  is a hole as well (the proof for the case that  $v_{i-1}$  is a hole is similar). If  $i + j$  is even, then the vertex  $v_{(i+j)/2}$  has equal distances to  $v_i$  and  $v_j$  and does not separate them, and  $v_{(i+j)/2} \in L$  as  $i < (i+j)/2 < j$ , and all vertices on the path between  $v_i$  and  $v_j$  are in  $L$  (as  $v_{i+1}, \dots, v_{j-1}$  is an anti-gap). As  $|L| = k$ , there are at most  $k - 1$  separations between  $v_i$  and  $v_j$ , a contradiction. If  $i + j$  is odd, then the vertex  $v_{(i+j+1)/2}$  satisfies  $i < (i+j+1)/2 < j$ , so  $v_{(i+j+1)/2} \in L$ , and it has equal distances to  $v_i$  and to  $v_{j+1}$ , and there are at most  $k - 1$  separations between these two holes, a contradiction again. ■

In the next three lemmas,  $L \subseteq V$  is a landmark set that satisfies  $|L| = k$ .

**Lemma 7** *If  $L$  induces at least two gaps, then the length of every anti-gap that does not contain an end vertex is even.*

**Proof.** Assume by contradiction that there is an anti-gap of odd length  $v_i, \dots, v_{i+2x}$  for some integer  $x \geq 1$ , where  $i > 1$  and  $i + 2x < n$ , such that  $v_{i-1}$  and  $v_{i+2x+1}$  are holes. The vertex  $v_{i+x}$  is a part of the anti-gap as  $i < i+x < i+2x$ , but it does not separate the holes  $v_{i-1}$  and  $v_{i+2x+1}$ , as it has equal distances of  $x + 1$  to both of them. Thus, there are at most  $k - 1$  separations between these anti-holes, a contradiction. ■

**Lemma 8** *If  $L$  induces at least three gaps, then all anti-gaps that do not contain end vertices have the same length.*

**Proof.** Assume by contradiction that there are two anti-gaps of different lengths that do not have end vertices. Thus, there are two such consecutive anti-gaps (with even numbers of vertices) where the first sequence is of the form  $v_i, \dots, v_{i+2x-1}$ , and the second sequence is of the form  $v_{i+2x+1}, \dots, v_{i+2x+2y}$ , such that  $v_{i-1}$ ,  $v_{i+2x}$ , and  $v_{i+2x+2y+1}$  are holes (where  $x, y$  are integers such that  $x, y \geq 1$  and  $x \neq y$ , as the anti-gaps have lengths  $2x$  and  $2y$ , that are distinct even integers). We claim that the vertex  $v_{i+x+y}$  is an anti-hole. The only hole among  $v_i, \dots, v_{i+2x+2y}$  is  $v_{i+2x}$  (obviously  $v_{i+x+y}$  is one of the vertices  $v_i, \dots, v_{i+2x+2y}$  as  $i < i+x+y < i+2x+2y$ ). The vertices  $v_{i+2x}$  and  $v_{i+x+y}$  are distinct as  $x \neq y$ . The distances of  $v_{i+2x+2y+1}$  and  $v_{i-1}$  from  $v_{i+x+y}$  are both equal to  $x + y + 1$ , and therefore there are at most  $k - 1$  separations between  $v_{i+2x+2y+1}$  and  $v_{i-1}$  in  $L$ , a contradiction. ■

**Lemma 9** *If  $L$  induces at least two gaps, then the set of the  $n - k$  holes is of the form*

$$v_i, v_{i+\rho}, v_{i+2\rho}, \dots, v_{i+(n-k-1)\rho}$$

where  $\rho \geq 3$  is odd,  $i \geq 1$ , and  $i + (n - k - 1)\rho \leq n$ . If  $G$  has a landmark set with  $k$  vertices that induces at least two gaps, then  $n \leq 3k/2 + 1$ .

**Proof.** As there are  $k$  anti-holes, there are  $n - k$  holes. Every gap consists of a single hole. Let  $i$  be the hole of smallest index. Let  $v_{i+\rho}$  be the next hole. The number  $\rho - 1$  must be a positive even number (by Lemma 7), and thus  $\rho \geq 3$ , and  $\rho$  is odd. All other anti-gaps excluding the ones that possibly occur before the first hole and after the last hole also have lengths of  $\rho - 1$  (by Lemma 8) and gaps consisting of a single hole separate them.

Next, we show the second claim. We have  $i + (n - k - 1)\rho \leq n$  while  $n - k - 1 \geq 1$ ,  $\rho \geq 3$  and  $i \geq 1$ , implying  $1 + 3(n - k - 1) \leq n$ , or alternatively,  $n \leq 3k/2 + 1$ . ■

We now state the algorithm for all values of  $k$  and  $n \geq k + 2$ . An algorithm for finding a minimum cardinality landmark set simply returns vertices  $v_1, \dots, v_k$ . In the weighted variant, three solutions are calculated, and the output is a minimum weight solution out of these three solutions. The first solution is a set of  $k + 1$  minimum weight vertices. This solution is computed in time  $O(n)$ . The second solution is a minimum weight solution out of those with a single gap. These are solutions of the form  $\{v_1, \dots, v_a, v_b, \dots, v_n\}$  for  $0 \leq a < b \leq n + 1$ , where  $a + (n - b + 1) = k$ , and these solutions can be enumerated in time  $O(n)$ . If  $n \leq 3k/2 + 1$ , solutions inducing at least two gaps are considered too. For any odd integer  $3 \leq \rho \leq \frac{n-1}{k-1}$ , we find the maximum weight of a sequence of the form  $v_i, v_{i+\rho}, v_{i+2\rho}, \dots, v_{i+(n-k-1)\rho}$  by testing all relevant values of  $i$ . This can be done in time  $O(n)$  for each value of  $\rho$ , and in time  $O(n^2)$  in total.

### 3 Complete wheel graphs

This section deals with the case where  $G$  is complete wheel graph, where  $V = C \cup \{h\}$  such that  $C = \{c_1, \dots, c_{n-1}\}$  is  $G$ 's cycle,  $h$  is the central vertex, and the other vertices are cycle vertices. For simplicity, we extend the definition of indices of cycle vertices as follows. The vertex  $c_i$  for  $i \geq n$  or  $i \leq 0$  is defined as  $c_i = c_b$ , for an integer  $b$  where  $1 \leq b \leq n - 1$  such that  $b = i + j(n - 1)$  for an integer  $j$  (where  $j$  may be positive or negative). In such a graph,  $E = \{\{c_i, h\} | 1 \leq i \leq n - 1\} \cup \{\{c_i, c_{i+1}\} | 1 \leq i \leq n - 1\}$ .

In wheel graphs, any distance between any possible pair of vertices is either 1 or 2. Distances of 1 occur only between pairs of vertices connected by an edge, that is, pairs of neighboring cycle vertices, and pairs consisting of  $h$  and a cycle vertex. Note that  $n \geq 4$ , since  $G$ 's cycle has at least 3 vertices. Note, also, that for a pair of vertices  $\{u, v\} \subset V$ ,  $N_{u,v}(G)$  is never empty as it contains  $h$  if  $u, v \neq h$ , and if  $\{u, v\} = \{h, c_i\}$ , then  $N_{u,v}(C) = \{c_{i-1}, c_{i+1}\}$  (however, in the case that we only consider neighbors on  $G$ 's cycle,  $C$ , it is possible that  $N_{u,v}(C) = \emptyset$ ). We start the discussion by examining small wheels; these cases are slightly different from larger wheels as there is no fixed pattern with respect to possible positions of landmarks. Afterwards, we discuss the more general case of larger wheels, where we will show that for  $n \geq 9$ , the central vertex can be omitted for any landmark set  $L \subseteq V$ , for both models (i.e. if  $L \in LS_k^M(G)$  then  $L \setminus \{h\} \in LS_k^M(G)$ ). Since for  $n = 4$ , a complete wheel graph is actually a complete graph (or a clique graph), we deal with the more general case of complete graphs first.

#### 3.1 Complete graphs (cliques) and complete bipartite graphs

Let  $G = (V, E)$  be a complete graph with  $n \geq 3$  (i.e.  $E = \{\{u, v\} | u, v \in V\}$ ). In this graph the distances satisfy  $d(u, v) = 1$  for any  $u, v \in V$ .

**Proposition 10** *For a complete graph  $G$ ,  $md_2^{AP}(G) = n$ , and  $md_k^{AP}(G) = \infty$  for  $k > 2$ . Additionally,  $md_k^{NL}(G) = n - 1$ . For any  $G$  and  $k$ , a minimum weight landmark set can be computed in linear time.*

**Proof.** For a pair of vertices  $u \neq v$ , any third vertex  $x \neq u, v$  does not separate them. Thus, a landmark set for any model must contain at least  $n - 1$  vertices. Since any subset of  $n - 1$  vertices is a trivial landmark set for NL, we find  $md_k^{NL}(G) = n - 1$ . Next, consider AP. Since for any pair of vertices  $u, v$ ,  $SpS_{u,v}(L) = \{u, v\} \cap L$ , at most two separations are possible, so  $md_k^{AP}(G) = \infty$  for  $k > 2$ , and for  $k = 2$ ,  $L$  must contain all vertices to allow two separations for every pair. We briefly discuss algorithms for minimum weight landmark sets. For AP, the algorithm returns  $V$  if  $k = 2$ , and otherwise it reports that there is no feasible solution. For NL, it finds a vertex  $y$  of maximum weight and returns  $V \setminus \{y\}$ . ■

Next, let  $G = (V, E)$  be a complete bipartite graph with  $n \geq 3$  with the partitions  $A, B$  (where  $A, B \neq \emptyset$ ,  $A \cup B = V$ ,  $A \cap B = \emptyset$  and  $E = \{\{u, v\} | u \in A, v \in B\}$ ). In this graph the distances satisfy  $d(u, v) = 1$  for  $u \in A$  and  $v \in B$ , or if  $u \in B$  and  $v \in A$  and  $d(u, v) = 2$  if  $u \neq v$ ,  $u, v \in A$  or  $u, v \in B$ .

**Proposition 11** *For a complete bipartite graph  $G$ ,  $md_2^{AP}(G) = n$ , and  $md_k^{AP}(G) = \infty$  for  $k > 2$ . Additionally,  $md_k^{NL}(G) = n - 2$  for  $n \geq k + 2$ . For any  $G$  and  $k$ , a minimum weight landmark set can be computed in linear time.*

**Proof.** For a pair of vertices  $u \neq v$  that belong to the same partition ( $A$  or  $B$ ), any third vertex  $x \neq u, v$  does not separate them. Thus, a landmark set for AP must contain all vertices proving the claim for AP. By the same reasoning, for NL, a landmark set must contain all vertices of  $A$  except for at most one vertex and all vertices of  $B$  except for at most one vertex. Given  $a \in A$  and  $b \in B$ ,  $V \setminus \{a, b\}$  is a landmark set since  $k \leq n - 2$ , every vertex of  $B$  (excluding  $b$ ) has distance 2 to  $b$  and distance 1 to  $a$ , and every vertex of  $A$  (excluding  $a$ ) has distance 1 to  $b$  and distance 2 to  $a$ . For NL, an algorithm finds a pair of vertices (one from  $A$  and one from  $B$  of maximum weights and returns  $V \setminus \{a, b\}$ . ■

### 3.2 General properties and the case $n = 5$

After covering the case of a complete graph, we get back to the case  $G$  is a complete wheel graph as described above, where  $n \geq 5$ . We continue with several simple properties that will be used throughout the section.

**Claim 12** *Let  $u, v \in G$  be cycle vertices. Then  $u$  and  $v$  can be separated by vertices in  $((N_u(G) \cup N_v(G)) \setminus N_{u,v}(G)) \cup \{u, v\}$ . Moreover,  $u$  and  $h$  can be separated by vertices in  $(C \setminus N_u(G)) \cup \{u, h\}$ .*

**Proof.** As all distances in the graph are in  $\{0, 1, 2\}$ , a vertex  $x \neq u, v$  that separates  $u$  and  $v$  has distance 1 to one of these vertices and distance 2 to the other. It is therefore a neighbor of one of them but not of the other, proving the first part. As the distance of  $h$  to any cycle vertex is 1, the only two cycle vertices that cannot separate  $h$  and  $u$  are the two neighbors of  $u$ . ■

We say that  $c_i$  and  $c_j$  are distant if both paths on the cycle between  $c_i$  and  $c_j$  are of length at least 3, that is, each such path has at least two cycle vertices between them. In this case  $j = i + \ell$  where  $3 \leq \ell \leq n - 4$ . If  $n \in \{5, 6\}$ , then no distant pairs of vertices exist.



**Corollary 13** *Let  $1 \leq i \leq n-1$ , and consider a set  $L \subseteq V$ . If  $n \geq 5$ , we have  $SpS(c_i, c_{i+1})(L) = L \cap \{c_{i-1}, c_i, c_{i+1}, c_{i+2}\}$ , if  $n \geq 6$ ,  $SpS(c_i, c_{i+2})(L) = L \cap \{c_{i-1}, c_i, c_{i+2}, c_{i+3}\}$ , and if  $n \geq 7$ , for any pair of distant vertices  $c_i, c_j$ ,  $SpS(c_i, c_j)(L) = L \cap \{c_{i-1}, c_i, c_{i+1}, c_{j-1}, c_j, c_{j+1}\}$ .*

**Proof.** We use Claim 12 for the three cases. ■

Next, we provide an analysis of the case  $n = 5$ . It is obvious that a minimum cardinality or minimum weight landmark set can be found in constant time by enumerating all subsets of vertices for constant values of  $n$ . In this section we will find all minimal landmark sets (with respect to set inclusion), which allows an algorithm to enumerate a smaller set of candidate subsets. In the next section this is done for  $6 \leq n \leq 8$  as well.

**Theorem 14** *Let  $n = 5$ .*

- *For AP,  $md_2^{AP}(G) = 4$ ,  $wmd_2^{AP}(G) = w(C)$ , and  $md_3^{AP}(G) = \infty$ .*
- *For NL,  $md_2^{NL}(G) = 3$ , and*

$$wmd_2^{NL}(G) = \min\{w(C), w(V) - \max_{1 \leq i \leq 4} (w(c_i) + w(c_{i+1}))\}.$$

*Additionally,  $md_k^{NL}(G) = 4$  for  $k \geq 3$ , and*

$$wmd_k^{NL}(G) = \min\{w(C), W(V) - \max_{1 \leq i \leq 4} w(c_i)\}.$$

**Proof.** First, consider AP and  $k = 2$ . A pair of cycle vertices  $c_i$  and  $c_{i+2}$  cannot be separated by any vertex except for  $c_i$  and  $c_{i+2}$ , as they have distances of 1 to all other vertices. Thus, all the cycle vertices (which can be split into two such pairs) must belong to any landmark set. We claim that  $C$  is indeed a landmark set (and it is a unique minimal landmark set). By Lemma 1, it is sufficient to show two separations between  $c_i$  (for  $1 \leq i \leq 4$ ) and  $h$ . One separation results from  $c_i$  being a member of the landmark set. Additionally,  $d(h, c_{i+2}) = 1$  while  $d(c_i, c_{i+2}) = 2$ , so  $c_{i+2}$  separates  $c_i$  and  $h$ . Note that  $V$  is a landmark set as well, but it is not minimal.

Next, consider NL and  $k = 2$ , and let  $L$  be a landmark set. In this case, for  $i = 1, 2$ , at least one of  $c_i$  and  $c_{i+2}$  must belong to  $L$  (otherwise they cannot be separated, as explained above). If there is at least one index  $j$  such that  $c_j \notin L$  and in addition,  $h \notin L$ , then as the distances of  $h$  and  $c_j$  are 1 both to  $c_{j-1}$  and to  $c_{j+1}$ , only  $c_{j+2}$  can separate them (if it is in  $L$ ), and  $L$  cannot be a landmark set. Thus, if  $h \notin L$ , then  $L = C$ , where  $C$  is a trivial landmark set. On the other hand, a set of the form  $\{c_i, c_{i+1}, h\}$  is a landmark set, as  $c_{i-1}$  and  $c_{i+2}$  are separated by both  $c_i$  and  $c_{i+1}$ . Therefore, a minimal landmark set (with respect to set inclusion) consists of either of  $C$ , or  $h$  and two adjacent cycle vertices.

Finally, we deal with  $k \geq 3$ . For AP, due to the above, there cannot be more than two separations between  $c_i$  and  $c_{i+2}$ , so  $md_k^{AP} = \infty$ . For NL and  $k \geq 4$ , a non-trivial landmark set must contain at least four vertices (otherwise there cannot be four separations), thus only trivial solutions exist. For NL and  $k = 3$ , as already for  $k = 2$  any landmark set with at most three vertices must contain  $h$  and two cycle vertices, for such a set, the other two cycle vertices will only have two separations. Thus, for  $k = 3$  only trivial solutions exist as well. ■

In what follows, we deal with the case  $n \geq 6$ . The crucial part of designing an algorithm for each option (where an option consists of the model and the value of  $k$ ) is to find a suitable condition

on the positions of landmarks on the cycle. In the case  $k = 1$ , given a landmark set  $L \subseteq V$ , let a gap be a maximum cardinality set of consecutive cycle vertices in  $V \setminus L$ . Two gaps are consecutive if they are separated by exactly one landmark (the set of vertices of the two gaps and the single landmark are consecutive vertices). The condition (defined for  $n \geq 8$  [10]) is as follows. There is no gap of four vertices or more, there is no pair of consecutive gaps such that both contain more than one vertex, and finally, there is at most one gap of three vertices. Here, the exact conditions for the two models are different from the condition for  $k = 1$ , and in fact we define four conditions for four cases. However, landmark sets for sufficiently large  $n$  ( $n \geq 9$ ) can be computed using one general approach for all cases (via dynamic programming).

### 3.3 All-pairs model (AP)

#### 3.3.1 The case $k = 2$

Consider the following condition for  $n \geq 6$ .

**Condition 1** *For any pair of consecutive vertices of  $C$  that are not in  $L$ , the three consecutive vertices of  $C$  preceding them are in  $L$ , and the three consecutive vertices of  $C$  following them are in  $L$ . That is, if  $c_i, c_{i+1} \notin L$ , then  $c_{i-3}, c_{i-2}, c_{i-1}, c_{i+2}, c_{i+3}, c_{i+4} \in L$ .*

Note that in particular, the condition implies that there does not exist a set of three or more consecutive vertices of  $C$  such that none of them is in  $L$ .

**Lemma 15** *For  $n \geq 6$ , if  $L \in LS_2^{AP}(G)$ , then Condition 1 holds.*

**Proof.** We show that the condition is necessary for a landmark set  $L$ . Assume that  $c_i, c_{i+1} \notin L$ . We show that  $c_{i+2}, c_{i+3}, c_{i+4} \in L$  (the proof for the vertices  $c_{i-3}, c_{i-2}, c_{i-1}$  is similar, and note that the two sets are not necessarily disjoint, and if  $n = 6$ , this is the same set). To obtain two separations between  $c_{i+1}$  and  $c_{i+2}$ , both  $c_{i+2}$  and  $c_{i+3}$  must be landmarks, and to obtain two separations between  $c_{i+1}$  and  $c_{i+3}$ ,  $c_{i+3}$  and  $c_{i+4}$  must be landmarks (since  $c_i$  and  $c_{i+1}$  are not landmarks). ■

We analyze the cardinality of sets that satisfy Condition 1.

**Proposition 16** *Any set satisfying Condition 1 has at least  $\lfloor \frac{n}{2} \rfloor$  cycle vertices. If  $L \subseteq V$  is a set where  $|L \cap C| \geq 4$ , then there are at least two separations in  $L \cap C$  for any pair  $h, c_i$ , and in this case, if  $L$  is a landmark set, then  $L \setminus \{h\}$  is a landmark set.*

**Proof.** We start with the first part. Consider a set  $X$  that satisfies the condition. If  $C \subseteq X$ , then we are done as  $n - 1 \geq \frac{n}{2}$  for  $n \geq 5$ . If  $X \cap C = \emptyset$ , then since  $|C| \geq 5$ , there are at least three consecutive cycle vertices not in  $X$ , and the condition is not satisfied. Otherwise, let  $c_{i+1}$  be a cycle vertex such that  $c_{i+1} \in X$  while  $c_i \notin X$ . Create a binary string of length  $n - 1$ , where the  $j$ th bit of the string is 1 if  $c_{i+j} \in X$  and 0 otherwise. The string starts with 1 and ends with 0. Partition the string into maximum length substrings that each substring starts with 1, ends with 0, and contains ones followed by zeroes. A substring never has more than two zeroes, and if it has two zeroes, then it has at least three ones preceding them. Thus, the number of ones is at least the number of zeroes, proving that there are at least  $\lceil \frac{n-1}{2} \rceil = \lfloor \frac{n}{2} \rfloor$  ones.

To prove the second part, we note that  $h$  can only separate a pair of vertices of the form  $h, c_i$ . As  $L \setminus \{h, c_{i-1}, c_{i+1}\}$  contains at least two cycle vertices, there are at least two separations between  $h$  and  $c_i$  even without  $h$ . ■

**Lemma 17** *Let  $n \geq 6$ . For a set  $L \subseteq C$ , if Condition 1 holds for  $L$ , then for any pair  $c_i, c_j$  there are two separations.*

**Proof.** Consider a set  $L$  satisfying the condition. We will show that any pair of cycle vertices has at least two separations. Consider a pair  $c_i, c_{i+\ell}$ , where  $\ell = 1$  or  $\ell = 2$ , and consider the list of vertices  $c_{i-1}, c_i, c_{i+\ell}, c_{i+\ell+1}$ . We show that  $L$  has at least two vertices out of this list. If  $L$  contains at least one vertex out of the first two vertices of the list, and at least one vertex of the last two vertices of the list, then we are done. If the first two vertices of the list are not in  $L$ , then by the condition, the vertices  $c_{i+1}, c_{i+2}, c_{i+3}$  must be in  $L$ , so the other two vertices of the list that can separate  $c_i$  and  $c_{i+\ell}$  are in  $L$ , and similarly, if  $c_{i+\ell}, c_{i+\ell+1}$  are not in  $L$ , then the other two vertices of the list are in  $L$ . This shows that in these cases, at least two out of the four vertices are in  $L$ . Consider a distant pair  $c_i, c_j$ . By the condition,  $c_{i-1}, c_i, c_{i+1}$  must have at least one vertex in  $L$ , and  $c_{j-1}, c_j, c_{j+1}$  must have at least one vertex in  $L$ , giving at least two separations between  $c_i$  and  $c_j$ . ■

Based on the above, we get the following.

**Corollary 18** *For  $n \geq 8$ ,  $L \subseteq C$  is a landmark set if and only if it satisfies Condition 1. Moreover,  $md_2^{AP} = \lfloor \frac{n}{2} \rfloor$ .*

**Proof.** A set that satisfies the condition has at least  $\lfloor \frac{n}{2} \rfloor \geq 4$  cycle vertices. Such a set is a landmark set; by Proposition 16, there are two separations for any pair of the form  $h, c_i$ , and by Lemma 17, any pair of cycle vertices also has at least two separations. By Lemma 15, any landmark set satisfies the condition. This proves the first part, and we find  $md_2^{AP} \geq \lfloor \frac{n}{2} \rfloor$ . The set  $L = \{c_i | 1 \leq i \leq n-1, i \text{ is odd}\}$  satisfies the condition (and thus it is a landmark set) and it has  $\lceil \frac{n-1}{2} \rceil = \lfloor \frac{n}{2} \rfloor$  vertices, proving that for  $n \geq 8$ ,  $md_2^{AP} \leq \lfloor \frac{n}{2} \rfloor$ . ■

Next, we discuss minimal landmark sets for  $n = 6, 7, 8$ . By corollary 18, for  $n = 8$ , such sets do not contain  $h$ , and the sets of cycle vertices that satisfy the condition have at least four (cycle) vertices. Any subset of at least five cycle vertices is a landmark set, but only those with two consecutive non-landmarks are minimal, with respect to set inclusion (sets of the form  $c_j, c_{j+1}, \dots, c_{j+4}$  for  $1 \leq j \leq 7$ ), since if there are no consecutive non-landmarks, there must be three consecutive landmarks  $c_{i-1}, c_i, c_{i+1}$ , and  $c_i$  can be omitted. Any subset of four cycle vertices for which there is no pair of consecutive non-landmarks satisfies the condition, and it has the form  $c_i, c_{i+1}, c_{i+3}, c_{i+5}$  for some  $1 \leq i \leq 7$ .

For  $n = 6, 7$ , sets a set satisfying Condition 1 has at least three cycle vertices, and we consider such sets. The next Corollary specifies all minimal landmark sets for  $n = 6, 7$ , where unlike the case  $n \geq 8$ , there exist such sets that contain  $h$ .

**Corollary 19** *For  $n = 6, 7$ , given  $A \subseteq C$ , if  $|A| \geq 4$ , then  $A$  is a landmark set. Assume that  $|X| = 3$ , and  $X$  satisfies Condition 1. The set  $Y = X \cup \{h\}$  is a landmark set, while  $X$  is not a landmark set.*

**Proof.** If  $n = 6, 7$ , any set with four cycle vertices satisfies the condition; for  $n = 6$ , there is only one cycle vertex not in  $A$ , and for  $n = 7$ , there are at most two cycle vertices not in  $A$ , and the condition is satisfied no matter whether they are adjacent or not. By Proposition 16, there are two separations for any pair of the form  $h, c_i$ . By Lemma 17, any pair of cycle vertices also has at least two separations.

The set  $X$  has two separations for any pair of cycle vertices, by Lemma 17, and in  $Y$ , a pair  $h, c_j$  is separated by  $h$  and  $X \setminus \{c_{j-1}, c_{j+1}\}$ , which must contain at least one vertex, so  $Y$  is a landmark set. For  $n = 7$ , the only form of a set  $X$  that satisfies the condition is  $c_i, c_{i+2}, c_{i+4}$ , since in the case of two consecutive vertices not in  $X$ , all remaining cycle vertices must be in  $X$ . For this set,  $h$  and  $c_{i+1}$  are only separated by  $c_{i+4}$ . For  $n = 6$ ,  $X$  can be of one of two forms. If  $X$  consists of  $c_i, c_{i+1}, c_{i+3}$ , then  $c_{i+2}$  and  $h$  are only separated by  $c_i$ . If  $X$  consists of  $c_i, c_{i+1}, c_{i+2}$ , then  $c_{i+1}, h$  are only separated by  $c_{i+1}$ . Thus,  $X$  is never a landmark set. ■

Let  $n \geq 9$ . For a landmark set (that does not contain  $h$ ), we define a cyclic binary string of length  $n - 1$  where similar to the proof of Proposition 16, a zero denotes a non-landmark and a 1 denote a landmark. A string of five bits is called *invalid* (for AP and  $k = 2$ ) if it contains at least four zeroes, or if it contains three zeroes and it is not equal to 01010. Other strings of five bits are called good.

**Lemma 20** *Let  $n \geq 9$ . A cyclic binary string represents a landmark set if and only if it does not contain any invalid five bit string as a substring.*

**Proof.** We start with showing that if the substring corresponding to a set  $X$  has an invalid substring, then  $X$  does not satisfy Condition 1, and therefore it is not a landmark set. We show that an invalid substring always has a pair of consecutive zeroes. This holds for 00000, and it holds for any string with four zeroes (as it only has one 1). The only string of length five and three zeroes that does not have two consecutive zeroes is 01010, which is a good substring. Since any invalid substring has at least one additional zero except for the pair of consecutive zeroes, this means that there are two cycle vertices  $c_i, c_{i+1} \notin X$  (corresponding to the pair of consecutive zeroes), such that one of  $c_{i-3}, c_{i-2}, c_{i-1}, c_{i+2}, c_{i+3}, c_{i+4}$  has a zero in the substring and thus it is not in  $X$ , violating Condition 1.

Assume now that a set  $Y \subseteq C$  does not satisfy Condition 1. So, there is a pair  $c_i, c_{i+1} \notin Y$  such that one of  $c_{i-3}, c_{i-2}, c_{i-1}, c_{i+2}, c_{i+3}, c_{i+4}$  is not in  $Y$ . If one of  $c_{i-3}, c_{i-2}, c_{i-1}$  is not in  $Y$ , then the substring for  $c_{i-3}, c_{i-2}, c_{i-1}, c_i, c_{i+1}$  has at least three zeroes, and the last two bits are zeroes. If one of  $c_{i+2}, c_{i+3}, c_{i+4}$  is not in  $Y$ , then the substring for  $c_i, c_{i+1}, c_{i+2}, c_{i+3}, c_{i+4}$  starts with two zeroes, and it has at least one additional zero. In both cases we find an invalid substring. ■

In the next section we state a dynamic programming formulation for computing  $wmd_2^{AP}$  for a wheel  $G$  with  $n \geq 9$  that is based on the cyclic binary string, and on the fact that including  $h$  in landmark sets is not necessary. The remaining cases for AP ( $k \geq 3$ ) are simpler and we discuss them now.

### 3.3.2 The case $k = 3$

Let  $k = 3$  and consider the following condition for  $n \geq 6$ .

**Condition 2** *For any vertex of  $C$  that is not in  $L$ , the four consecutive vertices of  $C$  preceding it, and the four consecutive vertices of  $C$  following it are in  $L$ . That is, if  $c_i \notin L$ , then*

$$c_{i-4}, c_{i-3}, c_{i-2}, c_{i-1}, c_{i+1}, c_{i+2}, c_{i+3}, c_{i+4} \in L.$$

**Lemma 21** *For  $n \geq 6$ , if  $L \in LS_3^{AP}(G)$ , then Condition 2 holds.*

**Proof.** We show that the condition is necessary for a landmark set  $L$ . Assume that  $c_i \notin L$ . We show that  $c_{i+1}, c_{i+2}, c_{i+3}, c_{i+4} \in L$  (the proof for  $c_{i-4}, c_{i-3}, c_{i-2}, c_{i-1}$  is similar). To obtain three separations between  $c_i$  and  $c_{i+\ell}$  for  $\ell = 1$  or  $\ell = 2$ , at least three of  $c_{i-1}, c_i, c_{i+\ell}, c_{i+\ell+1}$  must be in  $L$ , and thus at least two of  $c_i, c_{i+\ell}, c_{i+\ell+1}$  must be in  $L$ , that is (since  $c_i \notin L$ ),  $c_{i+\ell}, c_{i+\ell+1} \in L$  for  $\ell = 1, 2$ , proving  $c_{i+1}, c_{i+2}, c_{i+3} \in L$ . To obtain three separations between  $c_{i+1}$  and  $c_{i+3}$ , at least three of  $c_i, c_{i+1}, c_{i+3}, c_{i+4}$  must be in  $L$ , and since  $c_i \notin L$ ,  $c_{i+4} \in L$ . ■

We analyze the cardinality of sets that satisfy the condition.

**Proposition 22** *Any set satisfying the Condition 2 has at least  $\lfloor \frac{4n}{5} \rfloor$  cycle vertices. If  $L \subseteq V$  is a set where  $|L \cap C| \geq 5$ , then there are at least three separations in  $L \cap C$  for a pair  $h, c_i$ , and in this case, if  $L$  is a landmark set, then  $L \setminus \{h\}$  is a landmark set.*

**Proof.** We start with the first part. Consider a set  $X$  that satisfies Condition 2. If  $C \subseteq X$ , then we are done as  $n - 1 \geq \frac{4n}{5}$  for  $n \geq 5$ . If  $X \cap C = \emptyset$ , then since  $|C| \geq 5$ , there are at least two consecutive cycle vertices not in  $X$ , and the condition is not satisfied. Otherwise, let  $c_{i+1}$  be a cycle vertex such that  $c_{i+1} \in X$  while  $c_i \notin X$ . Create a binary string of length  $n - 1$  for  $X$ . The string starts with four times 1 and ends with 0. Partition the string into maximum length substrings that each substring starts with 1, ends with 0, and contains ones followed by zeroes. Every sequence of ones appears after a zero in this cyclic string, and therefore it contains at least four ones by the condition, while a substring never has more than one zero. Thus, the number of ones is at least four times the number of zeroes, proving that there are at most  $\lfloor \frac{n-1}{5} \rfloor$  zeroes (and we are done since  $n - 1 - \lfloor \frac{n-1}{5} \rfloor = \lfloor \frac{4n}{5} \rfloor$ ).

To prove the second part, as  $L \setminus \{h, c_{i-1}, c_{i+1}\}$  contains at least three cycle vertices, there are at least three separations between  $h$  and  $c_i$  even without  $h$ . ■

**Lemma 23** *Let  $n \geq 6$ . For a set  $L \subseteq C$ , if Condition 2 holds for  $L$ , then for any pair  $c_i, c_j$  there are at least three separations.*

**Proof.** Consider a set satisfying the condition. We will show that any pair of cycle vertices has at least three separations. Consider a pair  $c_i, c_{i+\ell}$ , where  $\ell = 1$  or  $\ell = 2$ , and consider the vertices  $c_{i-1}, c_i, c_{i+\ell}, c_{i+\ell+1}$ . By the condition, at most one of them is not in  $L$  (as a vertex not in  $L$  has at least four vertices in  $L$  before it and following it), giving three separations. Consider a distant pair  $c_i, c_j$ . By the condition,  $c_{i-1}, c_i, c_{i+1}$  must have at least two vertices in  $L$ , and  $c_{j-1}, c_j, c_{j+1}$  must have at least two vertices in  $L$ , giving at least four separations between  $c_i$  and  $c_j$ . ■

**Corollary 24** *For  $n \geq 7$ ,  $L \subseteq C$  is a landmark set if and only if it satisfies the condition. Moreover,  $md_3^{AP} = \lfloor \frac{4n}{5} \rfloor$ .*

**Proof.** If  $n \geq 7$ , then a set that satisfies the condition has at least  $\lfloor \frac{4n}{5} \rfloor \geq 5$  cycle vertices (using Proposition 22), there are at least three separations for any pair of the form  $h, c_i$ . By Lemma 23, any pair of cycle vertices also has at least two separations. Thus, a set satisfying the condition is a landmark set, and by Lemma 21, every landmark set satisfies the condition. This proves the first part, and we find  $md_2^{AP} \geq \lfloor \frac{4n}{5} \rfloor$ . The set  $L = \{c_i | 1 \leq i \leq n - 1, i \text{ is not divisible by } 5\}$  satisfies the condition and has  $\left\lceil \frac{4(n-1)}{5} \right\rceil = \lfloor \frac{4n}{5} \rfloor$  vertices. ■

For  $n = 7, 8$ , a minimal landmark set (with respect to set inclusion) consists of  $n - 2$  cycle vertices, and any subset of  $n - 2$  vertices is a landmark set. For  $n = 6$ , a set that satisfies the condition has at least four cycle vertices. We now specify the structure of minimal landmark sets for  $n = 6$ , and show these sets are exactly all subsets of five vertices.

**Corollary 25** *For  $n = 6$ , a set of four cycle vertices is not a landmark set. Any subset of  $V$  with five vertices is a landmark set.*

**Proof.** A set of four (consecutive) cycle vertices  $c_{i-2}, c_{i-1}, c_{i+1}, c_{i+2}$  only makes two separations for the following three pairs:  $h, c_{i-2}$ ,  $h, c_i$ ,  $h, c_{i+2}$ . Any set of five vertices gives another separation to each one of the pairs, as both  $h$  and  $c_i$  separate each one of the three pairs. ■

A string of five bits is called *invalid* (for AP and  $k = 3$ ) if it has at least two zeroes.

**Lemma 26** *Let  $n \geq 9$ . A cyclic binary string represents a landmark set if and only if it does not contain any invalid five bit string as a substring.*

**Proof.** If a substring has at least two zeroes, then each of the vertices corresponding to the zeroes does not have at least four consecutive ones before and after it. Condition 2 is not satisfied, and therefore, the corresponding set is not a landmark set. Assume now that a set  $Y \subseteq C$  does not satisfy the condition. There is a pair  $c_i, c_{i+\ell} \notin Y$  for some  $1 \leq \ell \leq 4$ . The substring for  $c_i, c_{i+1}, c_{i+2}, c_{i+3}, c_{i+4}$  has two zeroes, and therefore it is invalid. ■

In the next section we state a dynamic programming formulation for computing  $wmd_3^{AP}$  based on the cyclic binary string, and on the fact that including  $h$  in landmark sets is not necessary for  $n \geq 9$ .

### 3.3.3 The case $k \geq 4$

Finally, for  $k \geq 4$ , we can prove the following.

**Theorem 27** *For AP, and  $n \geq 6$ ,  $md_k^{AP} = \infty$  for  $k \geq 5$ . Additionally,  $md_4^{AP} = n - 1$  if  $n \geq 7$  (in this case  $wmd_4^{AP} = w(C)$ ), and  $md_4^{AP} = 6$  for  $n = 6$  (in this case  $wmd_4^{AP} = w(V)$ ).*

**Proof.** Since a pair of the form  $c_i, c_{i+1}$  can be separated by exactly four vertices, a landmark set for  $k = 4$  must contain all cycle vertices, and there is no valid landmark set for  $k \geq 5$ . Any cycle vertex except for  $c_{i+1}$  and  $c_{i-1}$  separates  $h$  and  $c_i$ , thus  $C$  is a landmark set for  $k = 4$  if  $n \geq 7$ . For  $n = 6$ , there are only three separations by cycle vertices between  $h$  and  $c_i$ , and therefore  $V$  is the only landmark set (as  $h$  adds a separation). ■

## 3.4 Non-landmarks model (NL)

### 3.4.1 The case $k = 2$

Consider the following condition for  $n \geq 6$ .

**Condition 3** *For any pair of consecutive vertices of  $C$  that are not in  $L$ , the two consecutive vertices of  $C$  preceding them, and the two consecutive vertices of  $C$  following them are in  $L$ . That is, if  $c_i, c_{i+1} \notin L$ , then  $c_{i-2}, c_{i-1}, c_{i+2}, c_{i+3} \in L$ .*

Similarly to AP, the condition implies that there does not exist a set of three or more consecutive vertices of  $C$  such that none of them is in  $L$ .

**Lemma 28** *For  $n \geq 6$ , if  $L \in LS_2^{NL}(G)$ , then Condition 3 holds.*

**Proof.** We show that the condition is necessary for a landmark set  $L$ . Assume that  $c_i, c_{i+1} \notin L$ . We show that  $c_{i+2}, c_{i+3} \in L$  (the proof for the vertices  $c_{i-1}, c_{i-2}$  is similar, and here the two sets are again not necessarily disjoint). Assume by contradiction that  $c_{i+2} \notin L$ . Then, only  $c_{i+3}$  can separate between  $c_{i+1}$  and  $c_{i+2}$ , a contradiction. Assume by contradiction that  $c_{i+3} \notin L$ . Then, only  $c_{i+4}$  can separate between  $c_{i+1}$  and  $c_{i+3}$ , a contradiction. ■

We analyze the cardinality of sets that satisfy the condition.

**Proposition 29** *Any set satisfying Condition 3 has at least  $\lfloor \frac{n}{2} \rfloor$  cycle vertices. If  $L \subseteq V$  is a set where  $|L \cap C| \geq 4$ , then there are at least two separations in  $L \cap C$  for a pair  $h, c_i$ , and in this case, if  $L$  is a landmark set, then  $L \setminus \{h\}$  is a landmark set.*

**Proof.** We start with the first part. Consider a set  $X$  that satisfies the condition. If  $C \subseteq X$ , then we are done as  $n - 1 \geq \frac{n}{2}$  for  $n \geq 5$ . If  $X \cap C = \emptyset$ , then since  $|C| \geq 5$ , there are at least three consecutive cycle vertices not in  $X$ , and the condition is not satisfied. Otherwise, let  $c_{i+1}$  be a cycle vertex such that  $c_{i+1} \in X$  while  $c_i \notin X$ . Once again we create a binary string of length  $n - 1$  that starts with 1 and ends with 0, and partition the string into maximum length substrings that each substring starts with 1, ends with 0, and contains ones followed by zeroes. A substring never has more than two zeroes, and if it has two zeroes, then it has at least two ones. Thus, the number of ones is at least the number of zeroes, proving that there are at least  $\lceil \frac{n-1}{2} \rceil = \lfloor \frac{n}{2} \rfloor$  ones.

To prove the second part, as  $h$  cannot separate cycle vertices, and as  $L \setminus \{h, c_{i-1}, c_{i+1}\}$  contains at least two cycle vertices, there are at least two separations between  $h$  and  $c_i$  even without  $h$ . ■

**Lemma 30** *Let  $n \geq 6$ . For a set  $L \subseteq C$ , if Condition 3 holds, there are at least two separations for any pair  $c_i, c_j \notin L$ .*

**Proof.** Consider a set satisfying the condition. We show that any pair of cycle vertices not in  $L$  has at least two separations. Consider a pair  $c_i, c_{i+\ell} \notin L$ , where  $\ell = 1$  or  $\ell = 2$ . If  $c_{i-1} \notin L$ , then by the condition for the two vertices  $c_{i-1}, c_i$ , we have  $c_{i+1}, c_{i+2} \in L$ , contradicting  $c_{i+\ell} \notin L$ . Similarly, if  $c_{i+\ell+1} \notin L$ , then by the condition for the two vertices  $c_{i+\ell}, c_{i+\ell+1}$ , we have  $c_{i+\ell-1}, c_{i+\ell-2} \in L$ , a contradiction as one of the last two vertices is the vertex  $c_i$ . We find  $c_{i-1}, c_{i+\ell+1} \in L$ , and they are distinct vertices separating  $c_i$  and  $c_{i+\ell}$ . Next, consider a distant pair  $c_i, c_j \notin L$ . By the condition,  $c_{i-1}, c_i, c_{i+1}$  must have at least one vertex in  $L$ , and  $c_{j-1}, c_j, c_{j+1}$  must have at least one vertex in  $L$ , giving at least two separations between  $c_i$  and  $c_j$ . ■

Based on the above, we get the following.

**Corollary 31** *For  $n \geq 8$ ,  $L \subseteq C$  is a landmark set if and only if it satisfies Condition 3. Moreover,  $md_2^{NL} = \lfloor \frac{n}{2} \rfloor$ .*

**Proof.** A set that satisfies the condition has at least  $\lfloor \frac{n}{2} \rfloor \geq 4$  cycle vertices. Such a set is a landmark set as by Proposition 29, there are two separations for any pair of the form  $h, c_i$ , and by Lemma 30, any pair of cycle vertices also has at least two separations. By Lemma 28, any landmark set satisfies the condition. This proves the first part, and we find  $md_2^{NL} \geq \lfloor \frac{n}{2} \rfloor$ , while the other inequality follows as there is a landmark set of this cardinality for AP. ■

The minimal landmark sets (with respect to set inclusion) for  $n = 8$  are not the same as for AP, since the condition is slightly weaker. All subsets of four cycle vertices where there are no pair of consecutive non-landmark (that were defined for AP) are still minimal landmark sets.

Minimal landmark sets with a pair of consecutive non-landmarks are possible too, and have the form  $c_i, c_{i+1}, c_{i+3}, c_{i+4}$  (for some  $1 \leq i \leq 7$ ).

For  $n = 6, 7$ , we consider sets with at least three cycle vertices.

**Corollary 32** *For  $n = 6, 7$ , given  $A \subseteq C$ , if  $|A| \geq 4$ , then  $A$  is a landmark set. Assume that  $|X| = 3$ , and  $X$  satisfies Condition 3. The set  $Y = X \cup \{h\}$  is a landmark set, while  $X$  is a landmark set only if  $n = 6$ , and it consists of three consecutive cycle vertices.*

**Proof.** The subsets of four vertices are landmark sets as they were proved to be landmark sets for AP.

For  $n = 7$ , the proof that any landmark set has at least four vertices is the same as for AP. For  $n = 6$ , the proof for a triple of cycle vertices that are not consecutive is the same as for AP. If  $X$  consists of  $c_i, c_{i+1}, c_{i+2}$ , then  $c_{i-1}, h$  are separated by  $c_{i+1}$  and  $c_{i+2}$ , while  $c_{i+3}, h$  are separated by  $c_i$  and  $c_{i+1}$ , and  $c_{i-1}, c_{i+3}$  are separated by  $c_i$  and  $c_{i+2}$ . ■

We find that a minimal landmark set (with respect to set inclusion) for  $n = 7$  is the same as for AP. For  $n = 6$ , a minimal landmark set consists of either three consecutive cycle vertices, or  $h$  together with three cycle vertices that are not consecutive.

A string of four bits is called *invalid* (for NL and  $k = 2$ ) if it contains at least three zeroes.

**Lemma 33** *Let  $n \geq 9$ . A cyclic binary string represents a landmark set if and only if it does not contain any invalid four bit string as a substring.*

**Proof.** Let  $\alpha$  be an invalid substring. In this case,  $\alpha$  has a pair of consecutive zeroes, and at least one additional zero. This means that there are two cycle vertices  $c_i, c_{i+1} \notin X$ , such that one of  $c_{i-2}, c_{i-1}, c_{i+2}, c_{i+3}$  has a zero corresponding to it in  $\alpha$ , thus it is not in  $X$ , violating Condition 3.

Assume now that a set  $Y \subseteq C$  does not satisfy the condition. Thus, there is a pair  $c_i, c_{i+1} \notin Y$  such that one of  $c_{i-2}, c_{i-1}, c_{i+2}, c_{i+3}$  is not in  $Y$ . If one of  $c_{i-2}, c_{i-1}$  is not in  $Y$ , then the substring for  $c_{i-2}, c_{i-1}, c_i, c_{i+1}$  has at least three zeroes. If one of  $c_{i+2}, c_{i+3}$  is not in  $Y$ , then the substring for  $c_i, c_{i+1}, c_{i+2}, c_{i+3}$  has three zeroes. In both cases we find an invalid substring. ■

In the next section we state a dynamic programming formulation for computing  $wmd_2^{NL}$  based on the cyclic binary string, and on the fact that including  $h$  in any landmark set is not necessary for  $n \geq 9$ .

### 3.4.2 The cases $k = 3, 4$

Let  $k \in \{3, 4\}$  and consider the following condition.

**Condition 4** *For any vertex of  $C$  that is not in  $L$ , the two consecutive vertices of  $C$  preceding it, and the two consecutive vertices of  $C$  following it are in  $L$ . That is, if  $c_i \notin L$ , then  $c_{i-2}, c_{i-1}, c_{i+1}, c_{i+2} \in L$ .*

**Lemma 34** *For  $n \geq 5$ , if  $L \in LS_k^{NL}(G)$  for  $k = 3$  or  $k = 4$ , then Condition 4 holds.*

**Proof.** We show that the condition is necessary for a landmark set  $L$ . Assume that  $c_i \notin L$ . We show that  $c_{i+1}, c_{i+2} \in L$  (the proof for  $c_{i-2}, c_{i-1}$  is similar). If  $c_{i+\ell} \notin L$  for  $i \in \{1, 2\}$ , then there can be at most two separations between  $c_i$  and  $c_{i+\ell}$  (those are  $c_{i-1}$  and  $c_{i+\ell+1}$ ). Thus, as there must be at least three separations for every pair of non-landmarks and  $c_i \notin L$ , we find that  $c_{i+1}, c_{i+2} \in L$ . ■

We analyze the cardinality of sets that satisfy the condition.



**Proposition 35** *Any set satisfying Condition 4 has at least  $\lfloor \frac{2n}{3} \rfloor$  cycle vertices. If  $L \subseteq V$  is a set where  $|L \cap C| \geq k + 2$ , then there are at least  $k$  separations in  $L \cap C$  for a pair  $h, c_i$ , and in this case, if  $L$  is a landmark set, then  $L \setminus \{h\}$  is a landmark set.*

**Proof.** We start with the first part. Consider a set  $X$  that satisfies the condition. If  $C \subseteq X$ , then we are done as  $n - 1 \geq \frac{2n}{3}$  for  $n \geq 5$ . If  $X \cap C = \emptyset$ , then since  $|C| \geq 5$ , there are at least two consecutive cycle vertices not in  $X$ , and the condition is not satisfied. Otherwise, let  $c_{i+1}$  be a cycle vertex such that  $c_{i+1} \in X$  while  $c_i \notin X$ . Create a binary string of length  $n - 1$  for  $X$ . The string starts with 1 and ends with 0. Partition the string into maximum length substrings that each substring starts with 1, ends with 0, and contains ones followed by zeroes. A substring never has more than one zero. Thus, the number of ones is at least twice the number of zeroes, proving that there are at most  $\lfloor \frac{n-1}{3} \rfloor = n - 1 - \lfloor \frac{2n}{3} \rfloor$  zeroes.

To prove the second part, as  $L \setminus \{h, c_{i-1}, c_{i+1}\}$  contains at least  $k$  cycle vertices, there are at least  $k$  separations between  $h$  and  $c_i$  even without  $h$ . ■

**Lemma 36** *Let  $n \geq 6$ . For a set  $L \subseteq C$ , if Condition 4 holds for  $L$ , then for any pair  $c_i, c_j$  there are at least four separations.*

**Proof.** Consider a set satisfying the condition. We will show that any pair of non-landmark cycle vertices has at least four separations. Consider a pair  $c_i, c_{i+\ell}$ , where  $\ell = 1$  or  $\ell = 2$ . By the condition, it is impossible that both are non-landmarks. Thus, we are left with the case of a distant pair  $c_i, c_j$ . By the condition,  $c_{i-1}, c_i, c_{i+1}$  must have at least two vertices in  $L$ , and  $c_{j-1}, c_j, c_{j+1}$  must have at least two vertices in  $L$ , giving at least four separations between  $c_i$  and  $c_j$ . ■

**Corollary 37** *For  $n \geq 9$ ,  $L \subseteq C$  is a landmark set if and only if it satisfies the Condition 4. Moreover,  $md_k^{NL} = \lfloor \frac{2n}{3} \rfloor$  for  $k = 3, 4$ . For  $n = 8$  and  $k = 3$ ,  $L \subseteq C$  is a landmark set if and only if it satisfies the condition. In addition,  $md_3^{NL} = 5$ .*

**Proof.** Let  $n \geq 9$ . A set that satisfies the condition has at least  $\lfloor \frac{2n}{3} \rfloor \geq 6$  cycle vertices. Such a set is a landmark set as by Proposition 35, there are at least four separations for any pair of the form  $h, c_i$ , and by Lemma 36, any pair of non-landmark cycle vertices also has at least two separations. By Lemma 34, any landmark set satisfies the condition. This proves the first part, and we find  $md_k^{NL} \geq \lfloor \frac{2n}{3} \rfloor$ , for  $k = 3, 4$ . The set  $L = \{c_i | 1 \leq i \leq n - 1, i \text{ is not divisible by } 3\}$  satisfies the condition and has  $\lceil \frac{2(n-1)}{3} \rceil = \lfloor \frac{2n}{3} \rfloor$  vertices.

If  $n = 8$  and  $k = 3$ , a set that satisfies the condition has at least five cycle vertices (using Proposition 35), and there are three separations for any pair of the form  $h, c_i$ . By Lemma 36, any pair of non-landmark cycle vertices also has three separations. The set  $L$  above (which is  $\{c_1, c_2, c_4, c_5, c_7\}$ ) is a landmark set with five vertices. ■

It is left to consider the cases  $n = 6, 7, 8$ ,  $k = 4$ , and  $n = 6, 7$ ,  $k = 3$ .

**Lemma 38** *For  $n = 6$  and  $k = 3, 4$ , only trivial landmark sets exist ( $md_4^{NL} = 5$ ).*

*For  $n = 7$  and  $k = 3$ ,  $L \subseteq V$  is a landmark set if and only if it satisfies Condition 4 and  $|L| \geq 5$  (thus  $md_3^{NL} = 5$ ). For  $n = 7$  and  $k = 4$ ,  $L \subseteq V$  is a landmark set if and only if it either satisfies the Condition 4 and in addition  $h \in L$ , or  $L = C$  (thus  $md_4^{NL} = 5$ ).*

*For  $n = 8$  and  $k = 4$ ,  $L \subseteq V$  is a landmark set if and only if it either satisfies Condition 4 and in addition  $h \in L$ , or it has at least six cycle vertices (thus  $md_4^{NL} = 6$ ).*

**Proof.** For  $n = 6$ , a set  $L$  that satisfies the condition has at least four cycle vertices. If  $c_i, h \notin L$ , then as  $c_{i-1}, c_{i+1}$  do not separate  $c_i$  and  $h$ , there are two separations between them. Thus,  $L$  must contain another vertex,  $|L| \geq 5$ , and it is a trivial landmark set.

For  $n = 7$ , a set  $L$  that satisfies the condition has at least four cycle vertices. If it has exactly four cycle vertices, then it has the form  $c_{i-2}, c_{i-1}, c_{i+1}, c_{i+2}$ . If  $h \notin L$ , then similarly to the case  $n = 6$ , there are two separations between  $c_i$  and  $h$ . On the other hand if  $h \in L$ , this is a landmark set for  $k = 3, 4$ . Assume now that  $h \notin L$ , and thus  $L$  must contain at least five cycle vertices. If  $c_i \notin L$ , then there are three separations between  $h$  and  $c_i$ , so  $L$  is a landmark set for  $k = 3$  but not for  $k = 4$ , and the only landmark set for  $k = 4$  not containing  $h$  is  $C$ .

For  $n = 8$  and  $k = 4$ , a set  $L$  that satisfies the condition has at least five cycle vertices. If  $h \in L$ , then it is a landmark set. Otherwise, by the condition, if  $c_i \notin L$ , then  $c_{i-1}, c_{i+1} \in L$ , and to obtain four separations between  $c_i$  and  $h$ ,  $L$  must contain  $C \setminus \{c_i\}$ , which results in a landmark set of this form. ■

A string of three bits is called *invalid* string (for NL and  $k = 3, 4$ ) if it has at least two zeroes.

**Lemma 39** *Let  $n \geq 9$ . A cyclic binary string represents a landmark set if and only if it does not contain any invalid three bit string as a substring.*

**Proof.** If a substring has at least two zeroes, then each of the vertices corresponding to them is a non-landmark that does not have at least two consecutive ones before and after it, thus the corresponding set is not a landmark set. Assume now that a set  $Y \subseteq C$  does not satisfy the condition. There is a pair  $c_i, c_{i+\ell} \notin Y$  for some  $1 \leq \ell \leq 2$ . The substring for  $c_i, c_{i+1}, c_{i+2}$  has two zeroes, and therefore it is invalid. ■

In the next section we state a dynamic programming formulation for computing  $wmd_3^{AP}$  based on the cyclic binary string, and on the fact that including  $h$  in any landmark set is not necessary.

In this case the dynamic programming will be sufficiently simple, and we will not use cyclic binary strings.

### 3.4.3 The case $k \geq 5$

For  $k \geq 5$ , we can prove the following.

**Theorem 40** *For NL and  $n \geq 6$ ,  $md_k^{NL} = n - 2$  if  $n \geq k + 4$ , and  $md_k^{NL} = n - 1$  if  $n \leq k + 3$ .*

**Proof.** Since a pair of cycle vertices  $c_i, c_j$  that are not landmarks can be separated by exactly four vertices, any landmark set must contain all cycle vertices except for at most one. Any set consisting of  $C \setminus \{c_i\}$  for some  $1 \leq i \leq n - 1$  is a landmark set if and only if  $h$  and  $c_i$  can be separated by  $k$  cycle vertices. The number of cycle vertices that can separate them is  $n - 4$  (all cycle vertices except for  $c_{i-1}, c_i, c_{i+1}$ ). Thus, if  $k \geq n - 3$ , only trivial landmark sets exist, and otherwise the minimal landmark sets consist of all cycle vertices except for one. ■

An algorithm for the weighted version acts as follows. If  $n \leq 8$ , then it tests all minimal landmark sets defined above, and returns a landmark set of minimum weight. For each  $n = 4, 5, 6, 7, 8$ , these minimal sets are described above. For  $n \geq 9$ , we showed that any minimal landmark set (with respect to set inclusion) does not contain  $h$ . Thus, we test all subsets of cycle vertices according to the relevant conditions.

### 3.5 Dynamic programming formulations for $n \geq 9$

The general structure of the dynamic programming formulations is as follows. For a specific variant, consider the list of good strings for it of length  $q$  (where the list of invalid strings of length  $q$  is its complement set). Let  $S$  be the set of strings of  $q - 1$  bits, each of which being a substring of a good string of  $q$  bits (each substring consists of the first  $q - 1$  bits or the last  $q - 1$  bits of at least one good  $q$  bit string). Recall that for all cases  $q \leq 5$ , and therefore  $|S| \leq 16$ . We will compute  $|S|^2$  functions  $F_{\bar{\gamma}}^{\bar{\beta}}(i)$ , for  $\bar{\beta}, \bar{\gamma} \in S$ , and integral  $i$  such that  $q - 1 \leq i \leq n$ , where  $F_{\bar{\gamma}}^{\bar{\beta}}(i)$  denotes the minimum weight of a valid string of length  $i$  corresponding to  $c_1, c_2, \dots, c_i$  that does not have an invalid substring (seeing it as a linear string, not as a cyclic string), that starts with the string  $\bar{\beta}$  and ends with the string  $\bar{\gamma}$ . Obviously, as we deal with a cycle (of the wheel) rather than a path, the string should be cyclic, and the substring of the last  $q - 1$  bits (corresponding to  $c_{n-q+1}, c_{n-q+2}, \dots, c_{n-1}$ ) concatenated with the first  $q - 1$  bits (corresponding to  $c_1, c_2, \dots, c_{q-1}$ ) cannot contain an invalid substring. For every pair  $\bar{\beta}, \bar{\gamma} \in S$ , we determine whether a cyclic string starting with  $\bar{\beta}$  and ending with  $\bar{\gamma}$  would contain an invalid string. Let  $\hat{S} = \{(\bar{\beta}, \bar{\gamma}) | \bar{\beta}, \bar{\gamma} \in S, \bar{\gamma} \circ \bar{\beta} \text{ does not contain an invalid string}\}$  (where  $\circ$  denotes concatenation). Since  $q - 1 \leq 4$  and we deal with  $n - 1 \geq 8$ , the first  $q - 1$  bits and the last  $q - 1$  bits together correspond to eight distinct vertices. We let  $F_{\bar{\gamma}}^{\bar{\beta}}(i) = \infty$  for all  $i$  if  $\bar{\gamma} \notin S$  or  $\bar{\beta} \notin S$ . Additionally, we let  $F_{\bar{\gamma}}^{\bar{\beta}}(q-1) = \infty$  if  $\bar{\beta} \neq \bar{\gamma}$ , and for  $\bar{\beta} \in S$ , such that  $\bar{\beta} = \beta_1, \beta_2, \dots, \beta_{q-1}$ ,  $F_{\bar{\beta}}^{\bar{\beta}}(q-1) = \sum_{i=1}^{q-1} \beta_i \cdot w(c_i)$  (this is the cost incurred by placing landmarks in the vertices whose bits are equal to 1).

The values of  $F_{\bar{\gamma}}^{\bar{\beta}}(i)$  for  $q \leq i \leq n - 1$ ,  $\bar{\gamma}, \bar{\beta} \in S$  where  $\bar{\gamma} = \gamma_1, \gamma_2, \dots, \gamma_{q-1}$  are defined as follows.  $F_{\bar{\gamma}}^{\bar{\beta}}(i) = \gamma_{q-1} \cdot w(c_i) + \min\{F_{\bar{\xi}}^{\bar{\beta}}(i-1), F_{\bar{\delta}}^{\bar{\beta}}(i-1)\}$ , where  $\xi$  consists of the sequence  $0, \gamma_1, \gamma_2, \dots, \gamma_{q-2}$  and  $\delta$  consists of the sequence  $1, \gamma_1, \gamma_2, \dots, \gamma_{q-2}$ . The output is a subset of vertices that is found via backtracking using the minimum term out of  $F_{\bar{\gamma}}^{\bar{\beta}}(n-1)$  for all  $\bar{\gamma}, \bar{\beta}$  such that  $(\bar{\beta}, \bar{\gamma}) \in \hat{S}$ . The running time is  $O(n)$  for all cases, as the number of functions is constant (in all cases in fact  $|S| \leq 11$ , and thus there are at most 121 functions).

Next, we state the sets  $S$  and  $\hat{S}$  for the four algorithms. In the case  $k = 3, 4$  and NL, by Lemma 39, good strings are three bit strings with at most one zero. Thus  $S = \{11, 10, 01\}$ , and  $\hat{S} = \{(11, 11), (11, 10), (11, 01), (10, 11), (10, 01), (01, 11)\}$ . In the case  $k = 3$  and AP, by Lemma 26, good strings are five bit strings with at most one zero. Thus  $S = \{1111, 1110, 1101, 1011, 0111\}$ , and

$$\hat{S} = \{(1111, \bar{\gamma}) | \bar{\gamma} \in S\} \cap \{((1110, 1111), (1110, 1101), (1110, 1011), (1110, 0111), (1101, 1111), (1101, 1011), (1101, 0111), (1011, 1111), (1011, 0111), (0111, 1111))\}.$$

In the case  $k = 2$  and NL, by Lemma 33, good strings are four bit strings with at most two zeroes. Thus  $S = \{111, 110, 101, 100, 011, 010, 001\}$ , and

$$\hat{S} = \{(\bar{\beta}, \bar{\gamma}) | \bar{\beta} \in \{110, 111\}, \bar{\gamma} \in S\} \cap \{(101, \bar{\gamma}) | \bar{\gamma} \in S \setminus \{100\}\} \cap \{(100, \bar{\gamma}) | \bar{\gamma} \in S \setminus \{100, 110, 010\}\} \cap \{(011, 011), (011, 101), (011, 110), (011, 111), (010, 101), (010, 111), (010, 011), (001, 011), (001, 111)\}.$$

In the case  $k = 2$  and AP, by Lemma 20, good strings are five bit strings with at most two zeroes, or the string 01010. Thus  $S$  consists of all four bit strings with at most two zeroes (11 strings, that is, the four bit strings that are not in  $S$  are  $\{0000, 0001, 0010, 0100, 1000\}$ ), and

$$\hat{S} = \{(\bar{\beta}, \bar{\gamma}) | \bar{\beta} \in \{0101, 0110\}, \bar{\gamma} \in \{0101, 0111, 1011, 1101, 1111\}\} \cap \{(\bar{\beta}, \bar{\gamma}) | \bar{\beta} \in \{1111, 1110\}, \bar{\gamma} \in S\}$$

$$\begin{aligned}
& \cap \{(1101, \bar{\gamma}) | \bar{\gamma} \in S \setminus \{1100\}\} \cap \{(\bar{\beta}, \bar{\gamma} | \bar{\beta} \in \{1010, 1011\}, \bar{\gamma} \in S \setminus \{1001, 1100\}\} \\
& \cap \{(0111, \bar{\gamma}) | \bar{\gamma} \in \{0101, 0111, 1011, 1101, 1110, 1111\}\} \cap \{(1001, \bar{\gamma}) | \bar{\gamma} \in \{0011, 0111, 1011, 1111\}\} \\
& \cap \{(1100, \bar{\gamma}) | \bar{\gamma} \in S \setminus \{0110, 1010, 1100, 1110\}\} \cap \{(0011, \bar{\gamma}) | \bar{\gamma} \in \{0111, 1111\}\} .
\end{aligned}$$

## References

- [1] R. Adar and L. Epstein. Work in progress. 2014.
- [2] L. Babai. On the order of uniprimitive permutation groups. *Annals of Mathematics*, 113(3):553–568, 1981.
- [3] Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffmann, M. Mihalák, and L. S. Ram. Network discovery and verification. *IEEE Journal on Selected Areas in Communications*, 24(12):2168–2181, 2006.
- [4] J. Cáceres, M. C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, and D. R. Wood. On the metric dimension of cartesian products of graphs. *SIAM Journal on Discrete Mathematics*, 21(2):423–441, 2007.
- [5] G. Chartrand, L. Eroh, M. A. Johnson, and O. R. Oellermann. Resolvability in graphs and the metric dimension of a graph. *Discrete Applied Mathematics*, 105(1-3):99–113, 2000.
- [6] G. Chartrand and P. Zhang. The theory and applications of resolvability in graphs: A survey. *Congressus Numerantium*, 160:47–68, 2003.
- [7] X. Chen and C. Wang. Approximability of the minimum weighted doubly resolving set problem. In Z. Cai, A. Zelikovsky, and A. G. Bourgeois, editors, *COCOON*, volume 8591 of *Lecture Notes in Computer Science*, pages 357–368. Springer, 2014.
- [8] V. Chvátal. Mastermind. *Combinatorica*, 3(3):325–329, 1983.
- [9] J. Díaz, O. Pottonen, M. J. Serna, and E. J. van Leeuwen. On the complexity of metric dimension. In L. Epstein and P. Ferragina, editors, *ESA*, volume 7501 of *Lecture Notes in Computer Science*, pages 419–430. Springer, 2012.
- [10] L. Epstein, A. Levin, and G. J. Woeginger. The (weighted) metric dimension of graphs: Hard and easy cases. In M. C. Golumbic, M. Stern, A. Levy, and G. Morgenstern, editors, *WG*, volume 7551 of *Lecture Notes in Computer Science*, pages 114–125. Springer, 2012.
- [11] F. Harary and R. Melter. The metric dimension of a graph. *Ars Combinatoria*, 2:191–195, 1976.
- [12] S. Hartung and A. Nichterlein. On the parameterized and approximation hardness of metric dimension. In C. Umans, editor, *Proc. of IEEE Conference on Computational Complexity (CCC)*, pages 266–276, 2013.
- [13] M. Hauptmann, R. Schmied, and C. Viehmann. Approximation complexity of metric dimension problem. *Journal of Discrete Algorithms*, 14:214–222, 2012.

- [14] D. Johnson. The NP-completeness column: an ongoing guide. *Journal of Algorithms*, 6(3):434–451, 1985.
- [15] S. Khuller, B. Raghavachari, and A. Rosenfeld. Landmarks in graphs. *Discrete Applied Mathematics*, 70(3):217–229, 1996.
- [16] R. A. Melter and I. Tomescu. Metric bases in digital geometry. *Computer Vision, Graphics, and Image Processing*, 25:113–121, 1984.
- [17] A. Sebö and E. Tannier. On metric generators of graphs. *Mathematics of Operations Research*, 29(2):383–393, 2004.
- [18] B. Shanmukha, B. Sooryanarayana, and K. S. Harinath. Metric dimension of wheels. *Far East Journal of Applied Mathematics*, 8(3):217—229, 2002.
- [19] P. J. Slater. Leaves of trees. *Congressus Numerantium*, 14:549–559, 1975.